tion properly describes disturbances which grow or decay spatially, whereas the other describes disturbances which grow or decay temporally. Most experimental studies of the falling film problem have assumed that the disturbances grow temporally, when in fact they appear to grow spatially in most experiments. This note then stresses that the experimenter must determine whether the disturbances in his flow geometry exhibit spatial, temporal, or perhaps mixed mode growth or decay. Clearly, it is preferable to analyze the data emanating from stability experiments by using the predictions appropriate to the type of growth or decay encountered in the particular flow geometry. This obviates the need to employ Equation (9) which is neither trivial to apply nor exact for all but very weakly amplified disturbances.

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NOTATION

H = basic film thickness
N_{Re} = Reynolds number $= \overline{U}H/\nu$
N_ζ = surface tension group $= (\sigma/\rho)(3/g\nu^4)^{1/3}$
t = dimensionless time $= t^*\overline{U}/H$
U = dimensionless velocity of basic flow $= (3/2)(1 - y^2)$
$\overline{U}$ = average velocity of basic flow
x = dimensionless streamwise coordinate $= x^*/H$
y = dimensionless cross-stream coordinate $= y^*/H$

Greek Letters

α = dimensionless complex wave number $= \alpha^*H$
α_i = dimensionless spatial amplification factor
α_r = dimensionless real wave number
θ = angle of inclination of the plane to the horizontal
ν = kinematic viscosity
ρ = density
σ = surface tension
ϕ = dimensionless amplitude of the stream function $= \phi^*/\overline{U}H$
ψ = dimensionless stream function $= \psi^*/\overline{U}H$
ω = dimensionless complex angular frequency $= \omega^*H/\overline{U}$
ω_i = dimensionless temporal amplification factor
ω_r = dimensionless real angular frequency

Superscripts

$'$ = derivative with respect to y
$*$ = dimensional quantity

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On the Applicability of the Linear Stability Theory

S. P. LIN

Clarkson College of Technology
Potsdam, New York 13676

Shuler and Krantz claim in the previous article that the predictions of the spatial and temporal formulations are nonequivalent and that the predictions of the spatial formulation are in closer agreement with the observed data. The present author cannot but disagree completely for the following reasons. First of all, it should be pointed out that the linear stability theory, be it based on temporal or spatial formulation, is valid only if the neglected nonlinearity remains small. Thus the linear theory can be used to predict only the onset of instability and its subsequent exponential growth of the disturbances as long as the neglected nonlinear terms remain small. Note that at the onset of instability, $\omega_i = o$ for the temporal case and $\alpha_i = o$ for the spatial case. Thus the linear theory remains valid over a time T or a distance X such that $[\exp(\omega_i T) - \exp(o)] << 1$ or $[\exp(-\alpha_i X) - \exp(o)] << 1$. Therefore, for $T = 0(1)$, $X = 0(1)$, the linear theory is valid only if $|\omega_i| << 1|$ or $|\alpha_i| << 1$, regardless of the values of α_r. Note that when $|\omega_i| << 1$ or $|\alpha_i| << 1$, the amplification rate of the disturbance, be it short shear waves or long gravity waves, remains small. Under this condition, Gaster (1962) proved rigorously that the temporal growth rate and the spatial growth rate are related by

$$\omega_i = -\alpha_i(\partial\omega_r/\partial\alpha_r) - \tfrac{1}{2}\alpha_i^2(\partial^2\omega_r/\partial\alpha_r\partial\alpha_i)$$

The last term in the above equation is omitted in the most general transformation of Gaster given by (9) of Shuler and Krantz. The above equation states that the two formulations give identical neutral curves and equivalent amplification rates. The above equivalence was rigorously proved, and therefore the eigenvalues obtained from the two different formulations must satisfy the above equations if the linear theory is applied to the situation where it is valid. No additional demonstration of equivalence is necessary, although the eigenvalues obtained from the two formulations are, in general, not the same in the region where the linear theory is invalid.

All previous workers, including Heisenberg (1924) who used the temporal formulation, need not be embarrassed. They should be congratulated because they used the theory which is capable of predicting the correct condition for the onset of instability and also the correct initial amplification rate which is simply related to the spatial growth rate by the Gaster transformation. Those workers who believe that spatial formulation of the linear stability theory is superior should be reminded of its shortcomings. Note that only one Fourier component of the disturbance is considered in either the temporal or spatial formulation of the theory. However, a spatially bounded but temporarily growing arbitrary disturbance $f(x, t)$ can be constructed by the Fourier super position

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i(\alpha x - \omega t)} d\alpha$$

where

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, t) e^{-i(\alpha x + \omega t)} dx$$

Since the Orr-Sommerfeld equation is linear, it can be shown easily that the eigenvalue, which gives the stability condition, for the above general disturbance is identical to that for a single Fourier component. Thus the usual normal mode approach of the stability theory is justifiable for the temporally growing disturbances. On the other hand, because of the exponential factor $\exp(-\alpha_i x)$, the Fourier integral of the spatially growing disturbance does not exist unless $\alpha_i \to 0$. Thus the normal mode approach is not justifiable for the case of spatial formulation except when $\alpha_i \to 0$. However, when $\alpha_i \to 0$, two formulations are equivalent as was shown in the previous paragraphs. In short, the two formulations are equivalent if the linear theory is applied to where it is valid. Finally, two formulations are truely nonequivalent when nonlinear effects are considered (Agrawal and Lin, 1975).

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A Mixture Theory Formulation for Particulate Sedimentation

A. BEDFORD and C. D. HILL

Department of Aerospace Engineering and Engineering Mechanics
The University of Texas at Austin
Austin, Texas 78712

The sedimentation, or settling, of one substance in solid particulate form through a second substance in liquid or gaseous form plays an important part in many chemical engineering processes. A large number of examples are cited by Zenz and Othmer (1960). Sedimentation is also of importance in medicine, where erythrocyte sedimentation has become a standard clinical test, and in meteorology.

Previous work on analyzing the sedimentation process typically has concentrated on developing empirical expressions for the terminal settling velocity of the particles in terms of the properties of the particles and fluid, and the volumetric concentration of the particles (Barnea and Mizrahi, 1973).

In the present work, the analysis of particulate sedimentation is approached within the framework of the continuum theory of mixtures (Bowen and Wiese, 1969; Bedford and Ingram, 1971) by regarding the particles and fluid as superimposed continua. The equations obtained are applicable to the analysis of transient one-dimensional sedimentation and, when specialized to the steady sedimentation of particles with uniform particle concentration, lend additional insight into the terminal settling velocity problem.

DEVELOPMENT

The problem considered is the sedimentation under gravity of solid particles of uniform composition, size, and shape through a fluid held in a container of constant cross section (Figure 1). The x axis denotes position, with the positive direction downward.

Regarding the particles and fluid as two superimposed continua, in the absence of reactions, dissolution, or other mass transfer processes, each constituent must satisfy the usual one-dimensional conservation of mass equation

$$\frac{\partial \hat{\rho}_p}{\partial t} + \frac{\partial}{\partial x}(\hat{\rho}_p U_p) = 0 \tag{1}$$

$$\frac{\partial \hat{\rho}_f}{\partial t} + \frac{\partial}{\partial x}(\hat{\rho}_f U_f) = 0 \tag{2}$$

Writing the partial mass densities $\hat{\rho}_p$ and $\hat{\rho}_f$ in terms of mass densities and volumetric concentrations as $\hat{\rho}_p = \phi_p \rho_p$ and $\hat{\rho}_f = \phi_f \rho_f$, and assuming incompressibility of the individual particles and the fluid so that ρ_p and ρ_f are constant, we can write Equations (1) and (2) as

$$\frac{\partial \phi_p}{\partial t} + \frac{\partial}{\partial x}(\phi_p U_p) = 0 \tag{3}$$

$$\frac{\partial \phi_f}{\partial t} + \frac{\partial}{\partial x}(\phi_f U_f) = 0 \tag{4}$$

Also, note that

$$\phi_p + \phi_f = 1 \tag{5}$$